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A one-parameter family of approximations of the solution of the initial Cauchy problem for the Dirac equation

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Abstract. A one-parameter family of approximations of the initial Cauchy problem for the Dirac equation in $(3+1)$ spacetime dimensions is given. The approximation is defined in terms of the mean translation operator $G(t, y)f = (4\pi\rho)^{-1}A\int_{|v|=\rho} gf(y-tv)\,\mathrm{d}S$. The family is indexed by the 'speed' ρ. The main result is that the approximation converges to the solution (in the L_2-norm) if $\rho \geqslant \sqrt{3}c$, c being the speed of light.

1. Introduction and summary

It is well known that Feynman and Hibbs approximated the solution of the one-dimensional Dirac equation by suitably defined integrals over the space of a polygonal path which a particle can follow in order to go from one point to another, which means that it moves backwards and forwards with constant speed. So this approximation has been constructed by means of the concept of 'particle'. However, if we consider the basis of quantum mechanics and the dual behaviour of matter, we can think of an approximation in terms of the concept of 'wave'. In fact, we interpret the approximation from [1] in this way when it is considered for one dimension. Unfortunately, this is not true in the multidimensional case. Nevertheless, it is true that the approximation can be expressed in terms of a superposition of functions of the form $f(x-tv)$, v being a vector in the three-dimensional space of a fixed length $|v| = \rho$. In the one-dimensional case ρ and v can be interpreted as wave speed and propagation direction, respectively. In addition, v takes the values ρ or $-\rho$, i.e. there are only two directions, which means the set of directions is naturally discrete. In contrast, a continuum of directions will be found in the three-dimensional case (of course, without the terms wave and propagation). Therefore, it must be decided if the set of directions is discrete or not. By the way, in the approximation (constructed from the formal solution of the Dirac equation obtained with the Fourier transform) from [1] only eight values of v were considered, namely their directions are determined by the centre of a cube and its vertices, and their length is $\rho = \sqrt{3}c$ (where c is the speed of light), i.e. the set of directions has been discretised. In contrast, in the approximation (whose construction is based on the expressions of the partial derivatives as an average of the directional derivatives) from [2] all values of v are permitted. Here, we have not opted to discretise it, as well as considering an arbitrary ρ. In summary, we give an approximation of the solution of the initial Cauchy problem for the Dirac equation in $3+1$ spacetime dimensions (see [1]). The approximation is in terms of suitably defined integrals over finite-dimensional spaces which can be interpreted as spaces of polygonal paths consisting of segments of arbitrary direction. All segments have a common fixed speed ρ. The approximation converges to the solution (in L_2-norm) if $\rho \geqslant \sqrt{3}c$, c being the speed of light.

Our approximation can be obtained by modifying that from [1] (see (20)). Fortunately, after modifying it, we have found that several definitions and some results are useful in developing this paper. In order to avoid redundancy, we do not reproduce them here (the reader can see them in [1]). Incidentally, we use the same notation as in [1]. On the other hand, large parts of some proofs are nearly identical with some parts of the proofs given in [1]. In view of this, we will only provide a sketch of those proofs here.

Notwithstanding the similarity between our work and that in [1], there are two important differences. To begin with, the approximation given in [1] involved paths whose segments were parallel to the axes of a fixed reference frame, while here the directions of segments are arbitrary. Secondly, we investigate a family of approximations indexed by the 'speed' ρ, which leads to the interesting conclusion that there is a minimum allowable speed, out to be greater than the speed of light. The work in [1] was restricted to one fixed value of ρ.

The existence of the minimum allowable segment speed, which equals $\sqrt{3}c$, was discovered by Jacobson (see [2]). However, we use a different approximation from that of Jacobson and again it turns out that the minimum speed is $\sqrt{3}c$. In addition, we find that our approximation can be expressed in terms of the amplification matrix defined by Jacobson (see equation (18) from [2]).

Our approximation depends on the electromagnetic (gauge) potentials which enter the formulae via the matrix value factor A given in equation (2.1) (the same factor appears in equation (20) from [1]). Nevertheless, it was suggested by Feynman (see [3]) that the potentials result in multiplication of each path C by a scalar (rather than a matrix) factor $\exp(-ie \int_C A_\mu(x(\tau)\, dx^\mu(\tau))$ (the notation was used by Jacobson in [2]). In our case, we deal with integrals over finite-dimensional spaces rather than integrals over spaces of paths. This causes an amplitude for each path not to be defined by our approximation, which should be considered as a mean amplitude for the set of all polygonal paths consisting of n segments. Thus, in an electromagnetic potential $a = (a_1, a_2, a_3)$ the mean amplitude for the set of all polygonal paths consisting of n segments is multiplied by a factor A. This factor can be expressed only in terms of the combination $\Sigma_{k=1}^{3} \alpha^k a_k$ for a short 'time' t.

2. Definition of the approximation $E(\pi, \tau)$

In [1] Suárez has expressed the operator $G(t, \cdot)$ as the mean of translation operators (see equations (1) and (20)). These translations were parallel to the axes of the fixed frame of reference (therefore, there are eight of these operators). We now define for each positive number ρ the operator $G(t, \cdot)$ in a space of continual functions of $\mathbb{R}^3$ by

$$G(t, y)f = (4\pi\rho)^{-1} A \int_{|v|=\rho} gf(y - tv)\, dS \tag{2.1}$$

where

$$A = A(t, y) = \exp(-it\phi(y)) \exp(-ibt\beta) \prod_{k=1}^{3} \exp(ita_k(y)\alpha_\kappa)$$

(see equation (20) from [1]), dS is the area element of the sphere of radius ρ, the matrix valued function g is defined by

$$g := \prod_{k=1}^{3} (I - \delta\rho^{-1} v_k \alpha^k) \tag{2.2}$$

where I is the 4×4 identity matrix and δ is a positive number which satisfies $\delta\rho = 3$, v_k, $k = 1, 2, 3$ are coordinates of the vector v of $\mathbb{R}^3$, which is defined by

$$v = \rho(\cos\phi, \sin\phi\cos\theta, \sin\phi\sin\theta) \qquad \theta\in[0, 2\pi], \phi\in[0, \pi] \tag{2.3}$$

and α^k, $k = 1, 2, 3$, are 4×4 Hermitian matrices which satisfy equation (2) from [1].

Note that the operator $G(t, \cdot)$ is again a mean of translation operators, but the directions of these translations are now arbitrary.

In addition, we can write

$$g = \prod_{k=1}^{3} (I - \delta\rho^{-1} v e_k \alpha^k) \tag{2.4}$$

where the vectors e_k, $k = 1, 2, 3$, form the usual basis for $\mathbb{R}^3$.

We define the approximation $E(\pi, \tau)$ on $C^0(\mathbb{R}^3)$ as in formula (3) from [1], namely

$$E(\pi, \tau)f(y) := G(\tau - t_k, y)G(t_k - t_{k-1}, x^k)\dots G(t_1 - t_0, x^1)f \tag{2.5}$$

where $\pi = \{t_0, t_1, \dots, t_l\}$ is a partition of $[0, s]$, $\tau\in\,]t_k, t_{k+1}]$ and $f\in C^0(\mathbb{R}^3)$.

In this paper we denote the inverse, the adjoint and the norm of a matrix T by T^{-1}, T^* and $|T|$, respectively. The latter is defined by $|T| = \sup_{|z|=1}|Tz|$ where $|z|$ denotes the Euclidean norm of vector $z\in\mathbb{C}^4$. We will employ the usual notation for the set of bounded operators on L_2, namely $B(L_2)$. Given $T\in B(L_2)$, the standard norm operator of T is defined and denoted by

$$\|T\| = \sup_{\|f\|=1} \|Tf\| \tag{2.6}$$

where $\|f\|$ denotes the L_2-norm of f.

Remark 2.1. Note that, if T is a 4×4 matrix, the function defined by $(Tf)(y) := Tf(y)$, $y\in\mathbb{R}^3$, since $\|Tf\| \leqslant |T|\|f\|$, belongs to L_2, if $f\in L_2$, implies that the map $f\mapsto Tf$, which we still denote by T, belong to $B(L_2)$ and its operator norm is not greater than $|T|$. It is not hard to prove $\|T\| = |T|$.

Remark 2.2. Note that, if $f\in C^0(\mathbb{R}^3)$ has support contained in Ω, then $G(t, \cdot)f$ has support contained in the neighbourhood $B(\Omega, t)$ of Ω. Consequently, $E(\pi, \tau)f$ has its support contained in the neighbourhood $B(\Omega, \tau)$ of Ω.

Remark 2.3. Note that, since ϕ, a_k, $k = 1, 2, 3$, are real valued functions, the hermiticity of α^k, $k = 1, 2, 3$, implies that $|Az| = |z|$ for all $z\in\mathbb{C}^4$. Here $|\,|$ denotes the Euclidian norm of $\mathbb{C}^4$.

3. Properties of the operators $G(t, \cdot)$ and $E(\pi, \tau)$

In this section, we need some previous definitions and notations. To begin with, B^n is the matrix-valued function defined by the 4×4 matrices α^k:

$$B^n(p) := \sum_{l_1<\dots<l_n} p_{l_1}\dots p_{l_n}\alpha^{l_1}\dots\alpha^{l_n} \qquad n = 0, 1, 2, 3, p\in\mathbb{R}^3 \tag{3.1}$$

where $B^0(p) := I =$ identity 4×4 matrix, $B^1(p) := \Sigma_{l=1}^3 p_l \alpha^l$. B_t, $|B|_t$ are the functions defined by the 'velocity' ρ, spherical Bessel functions of the first kind

$$\begin{aligned} &j_0(z) = \sin z/z \qquad j_1(z) = (\sin z - z\cos z)/z^2 \\ &j_2(z) = [(3-z^2)\sin z - 3z\cos z]/z^3 \\ &j_3(z) = [(15-6z^2)\sin z - (15-z^3)\cos z]/z^4 \end{aligned} \tag{3.2}$$

and evaluated at time t:

$$B_t(p) := \sum_{n=0}^{3} (-\mathrm{i}\delta)^n j_n(t\rho|p|) B^n(\hat{p}) \tag{3.3}$$

$$|B|_t^2(p) := \sum_{n=0}^{3} \delta^{2n} j_n^2(t_\rho|p|)(|B|^n(\hat{p}))^2 \qquad p \in \mathbb{R}^3 \tag{3.4}$$

where $\hat{p} := p/|p|$, and $|B|^n$, $n = 0, \ldots, 3$, are defined by

$$\begin{aligned} &(|B|^n(p))^2 := \sum_{l_1<\ldots<l_n}^{3} p_{l_1}^2 \cdots p_{l_n}^2 \qquad p \in \mathbb{R}^3 \\ &|B|^0(p) = 1 \qquad |B|^1(p) = |p|. \end{aligned} \tag{3.5}$$

Here, p_l, $l = 1, 2, 3$, are the coordinates of the vector p and, finally, $\langle\,,\rangle$ denotes the Euclidian inner product of $\mathbb{C}^4$.

Remark 3.1. It is clear that $|B|^n(p) \leqslant |p|^n/\sqrt{n!}$, $n = 0, 1, 2, 3$. Consequently,

$$|B|^n(\hat{p}) \leqslant 1/\sqrt{n!} \quad \text{for all } p \in \mathbb{R}^3. \tag{3.6}$$

The operator $G(t, \cdot)$ given in [1] turned out to be an isometry as well as the approximation (see lemma 1 from [1]). In our case, $G(t, \cdot)$ does not have this property. Nevertheless, we will be able to show the following lemma.

Lemma 3.1.

$$\|G(t,\cdot)f\| = \| |B|_t \hat{f} \| \qquad f \in L_2 \tag{3.7}$$

where $\hat{f}$ denotes the Fourier transform of f (see (7) from [1]).

The latter is a consequence of the next result.

Lemma 3.2.

$$G(t,\cdot)f = A\mathscr{F}^{-1}\left(\left[\sum_{n=0}^{3} (-\mathrm{i}\delta)^n j_n(t\rho|p|) B^n(p)\right]\hat{f}\right) \qquad f \in S(\mathbb{R}^3) \tag{3.8}$$

where $\mathscr{F}^{-1}$ and $S(\mathbb{R}^3)$ denotes the inverse of the Fourier transform (see (8) from [1]) and the space of the functions of rapid decrease, respectively.

We can now proceed to prove the formula (3.7).

Proof of lemma 3.1 [1]. Fix $f \in L_2$. Let $t \in \mathbb{R}^+$. Then, by (3.3) and (3.8), obtain that $\|G(t,\cdot)f\| = \|\mathscr{F}^{-1}(B_t(p)\hat{f}(p)\|$. Now by means of the Plancherel theorem (see proposition 1 from [1]) and (3.3), we prove formula (3.7).

Remark 3.2. Note that formula (3.8) permits us to extend the operator $G(t,\cdot)$ over L_2. Consequently, formula (3.7) is true for $f \in L_2$.

Remark 3.3. Note that, since $A(s,p)$ is an isometry in $\mathbb{C}^4$ for each $s \in \mathbb{R}^+$ and $p \in \mathbb{R}^3$, lemma 3.2, together with (3.3), implies that $\|G(t,\cdot)G(s,\cdot)f\| = \|G(t,\cdot)A^{-1}(s,\cdot)G(s,\cdot)f\| = \| |B|_t |B|_s \hat{f}\|$. If ϕ, a_k are constant functions, then we can continue this process to obtain, from (2.5), that

$$\|E(\pi,\tau)f\| = \| |B|_{\tau - t_k} |B|_{t_k - t_{k-1}} \ldots |B|_{t_1 - t_0} \hat{f}\|. \tag{3.9}$$

Now, we will give a proof of lemma 3.2. The next observations and results will lead us to it.

On account of the hermiticity of α^k, since $(\alpha^k)^2 = I$ (see (2) from [1]), we have that

$$(\alpha^{l_1} \ldots \alpha^{l_n})^*(\alpha^{k_1} \ldots \alpha^{k_m}) + (-1)^{m+n}(\alpha^{k_1} \ldots \alpha^{k_m})^*(\alpha^{l_1} \ldots \alpha^{l_n}) = 2\delta_{mn}\delta_{j_1 k_1} I$$

where $l_1 < \ldots < l_n$, $k_1 < \ldots < k_n$, $m, m = 1, 2, 3$. These results, together with (3.1), imply that

$$\begin{aligned}(B^n(p))^* B^m(p) &+ (-1)^{m+n}(B^m(p))^* B^n(p) \\ &= 2\delta_{nm} |B|^n(p)|B|^m(p) \qquad p \in \mathbb{R}^3,\ n, m = 0, 1, 2, 3.\end{aligned} \tag{3.10}$$

The next result justifies the notation in (3.4).

Proposition 3.1. If a_0, a_1, a_2, a_3 are complex numbers satisfying

$$\bar{a}_m a_n + (-1)^{m+n-1} a_m \bar{a}_n = 0 \qquad \text{for } m, n = 0, \ldots, 3 \tag{3.11}$$

then

$$\left|\left(\sum_{n=0}^{3} a_n B^n(p)\right) q\right|^2 = \left(\sum_{n=0}^{3} |a_n|^2 (|B|^n(p))^2\right) |q|^2 \qquad p \in \mathbb{R}^3,\ q \in \mathbb{C}^4. \tag{3.12}$$

Here, the overbar denotes the complex conjugate.

Proof. Fix $p \in \mathbb{R}^3$. Let $q \in \mathbb{C}^4$. Observe that

$$\left|\left(\sum_{n=0}^{3} a_n B^n(p)\right) q\right|^2 = \left\langle \left(\sum_{m=0}^{3} a_m B^m(p)\right) q, \left(\sum_{n=0}^{3} a_n B^n(p)\right) q \right\rangle$$

where $\langle , \rangle$ denotes the Euclidian inner product of $\mathbb{C}^4$. Now, by means of the definition of an adjoint matrix, and (3.10), it follows that

$$\begin{aligned}\left|\left(\sum_{n=0}^{3} a_n B^n(p)\right) q\right|^2 &\\ &= \left\langle \left(\sum_{m=0}^{3} a_n B^n(p)\right)^* \left(\sum_{m=0}^{3} a_m B^m(p)\right) q, q \right\rangle \\ &= \left\langle \left(\sum_{n=0}^{3} |a_n|^2 (|B|^n(p))^2 + \sum_{n<m} (\bar{a}_m a_n + (-1)^{m+n-1} a_m \bar{a}_n) \right.\right. \\ &\qquad \left.\left. \times (B^n(p))^*(B^m(p))\right) q, q \right\rangle.\end{aligned}$$

Thus, we obtain the formula (3.12) from (3.10).

Remark 3.4. The numbers $a_n = (-\mathrm{i}\delta)^n j_n(w)$, $n = 0, \ldots, 3$, $w \in \mathbb{R}^+$, satisfy the relationship (3.11). Thus, in view of (3.12), we obtain

$$|B_t(p)| = |B|_t(p) \qquad p \in \mathbb{R}^3,\ t \in \mathbb{R}^+. \tag{3.13}$$

The next result is an approach to lemma 3.2.

Proposition 3.2.

$$G(t,\cdot)f = (4\pi\rho)^{-1} A \mathscr{F}^{-1}\left[\left(\int_{|v|=\rho} \exp[-\mathrm{i}(vp)t] g(v)\, \mathrm{d}S\right) \hat{f}(p)\right] \qquad f \in S(\mathbb{R}^3). \tag{3.14}$$

Proof. On account of the Plancherel theorem and formula (2.4) we obtain

$$G(t,y)f = (4\pi\rho)^{-1}(2\pi)^{3/2} A \int_{|v|=\rho} \mathrm{d}S \int_{\mathbb{R}^3} \exp[\mathrm{i}(y - tv)p] g(v) \hat{f}(p)\, \mathrm{d}p.$$

Interchanging the integrals in this expression, by means of the definition of the Fourier transform of f (see (7) from [1]), we get formula (3.14).

To obtain formula (3.8), we need to calculate the integral $\int_{|v|=\rho} \exp(-\mathrm{i}tvp) g(v)\, \mathrm{d}S$. To perform this calculation, we replace vp by $(Tv)p$, where $T: \mathbb{R}^3 \mapsto \mathbb{R}^3$ is a linear transformation. We choose T to have the following properties: T is an isometry of $\mathbb{R}^3$ (with Euclidian norm); the absolute value of the determinant of T equals one ($|\det T| = 1$) and the inverse linear transform of T equals the adjoint linear transformation of T ($T^{-1} = T^*$). Note that the latter implies the other properties.

By means of the properties of T, we get

$$\begin{aligned}\int_{|v|=\rho} \exp(-\mathrm{i}tvp) g(v)\, \mathrm{d}S &= \int_{|Tv|=|v|=\rho} |\det T| \exp(-\mathrm{i}tpTv) g(Tv)\, \mathrm{d}S \\ &= \int_{|v|=\rho} \exp(-\mathrm{i}tvT^*p)\, \mathrm{d}S \\ &= \int_{|v|=\rho} \exp(-\mathrm{i}tvT^{-1}p) g(Tv)\, \mathrm{d}S.\end{aligned}$$

Now if, in addition, T fulfils $T(e_1) = \hat{p}$, from (2.3), it follows

$$\int_{|v|=\rho} \exp(-tvp) g(v)\, \mathrm{d}S = \int_{|v|=\rho} \exp(-\mathrm{i}t\rho|p|\cos\phi) g(Tv)\, \mathrm{d}S. \tag{3.15}$$

Our calculations now depend on the definition of g.

By means of (2.2) and (3.15), we obtain

$$\int_{|v|=\rho} \exp(-\mathrm{i}tvp) g\, \mathrm{d}S = \sum_{n=0}^{3} (-\delta\rho^{-1})^n \int_{|v|=\rho} \exp(-\mathrm{i}t\rho|p|\cos\phi) B^n(Tv)\, \mathrm{d}S. \tag{3.16}$$

Next, we express the integrals on the right-hand side of (3.16) in terms of spherical Bessel functions of the first kind. To do this, we regard the relationship

$$j_n(z) = (2z/\pi)^{-1/2} j_{n+1/2}(z) \tag{3.17}$$

where $j_{n+1/2}(z)$ denotes a Bessel function of the first kind of order $n+\frac{1}{2}$ (see, e.g. [4], p 54]). The latter admits the following integral representation (see, e.g., [4], p 48 equation (6)]):

$$j_{n+1/2}(z) = [(\pi z/2)^{1/2}(z/2)^n/n!] \int_0^\pi \exp(-\mathrm{i}z\cos\phi)\sin 2n+1\,\phi\,\mathrm{d}\phi. \tag{3.18}$$

In view of (3.5) and (3.18), since $T(e_1)=\hat{p}$, it follows

$$\begin{aligned}
&\int_{|v|=\rho} \exp(-\mathrm{i}\rho|p|t\cos\phi)B^0(Tv)\,\mathrm{d}S \\
&\qquad = \left(2\pi\rho\int_0^\pi \exp(-\mathrm{i}\rho|p|t\cos\phi)\sin\phi\,\mathrm{d}\phi\right) I \\
&\qquad = 4\pi\rho j_0(\rho|p|t)B^0(\hat{p}) \\
&\int_{|v|=\rho} \exp(-\mathrm{i}\rho|p|t\cos\phi)B^1(Tv)\,\mathrm{d}S \\
&\qquad = B^1\left[T\left(\int_{|v|=\rho}\exp(-\mathrm{i}\rho|p|t\cos\phi)v\,\mathrm{d}S\right)\right] \\
&\qquad = B\left(2\pi\rho^2\int_0^\pi \exp(-\mathrm{i}\rho|p|t\cos\phi)\cos\phi\sin\phi\,\mathrm{d}\phi T(e_1)\right) \\
&\qquad = \rho(4\pi\rho)j_1(\rho|p|t)B^1(\hat{p}).
\end{aligned} \tag{3.19}$$

In order to simplify the notation we denote $\int_{|v|=\rho}\exp(-\mathrm{i}\rho|p|t\cos\phi)v_{l_1}\dots v_{l_n}\,\mathrm{d}S$ by $v_{l_1}\dots v_{l_n}$.

By means of (3.5), the linearity of T and the inner product imply that

$$\begin{aligned}
&\int_{|v|=\rho}\exp(-\mathrm{i}\rho|p|t\cos\phi)B^n(Tv)\,\mathrm{d}S \\
&\qquad = \sum_{k_1<\dots<k_n}\left(\int_{|v|=\rho} e_{k_1}T_v\dots e_{k_n}Tv\,\mathrm{d}S\right)\alpha^{k_1}\dots\alpha^{k_n} \\
&\qquad = \sum_{k_1<\dots<k_n}\sum_{l_1,\dots,l_n=1} v_{l_1}\dots v_{l_n}\,e_{k_1}T(e_{l_1})\dots e_{k_n}T(e_{l_n}).
\end{aligned} \tag{3.20}$$

It is straightforward to verify that

$$v_1^{n-2}v_l^n = v_1^{n-2}v_3^n \qquad n=2,3 \qquad l=1,2. \tag{3.21}$$

In addition, an integration by parts proves

$$v_1^{n-2}[v_1^2-(n!/2)v_3^2] = (4\pi\rho)\rho^n j_n(\rho|p|t) \qquad n=2,3. \tag{3.22}$$

In other cases, the integrals equal zero.

Taking into account the latter results, we get

$$\int_{|v|=\rho} \exp(-\mathrm{i}\rho|p|t\cos\phi)B^2(Tv)\,\mathrm{d}S = \sum_{k_1<k_2}^{3}\left(v_1^2 e_{k_1}T(e_1)e_{k_2}T(e_1) + v_3^2\sum_{l=2}^{3} e_{k_1}T(e_1)e_{k_2}T(e_1)\right)$$

$$\int_{|v|=\rho} \exp(-\mathrm{i}\rho|p|t\cos\phi)B^3(Tv)\,\mathrm{d}S \tag{3.23}$$

$$= \sum_{k_1<k_2<k_3}\left(v_1^3 e_{k_1}T(e_1)e_{k_2}T(e_1)e_{k_3}T(e_1) + v_1v_3^2\sum_{m=1}^{3}\sum_{l=2}^{3} e_{k_m}T(e_1)\prod_{\substack{n=1\\ n\neq m}}^{3} e_{k_n}T(e_1)\right).$$

Now, we completely determine the linear transformation T. We only need to define $T(e_2)$ and $T(e_3)$ (see equation (3.15)). To begin with, we choose the vectors $T(e_2)$ and $T(e_3)$ to be unitary ($|T(e_2)|=|T(e_3)|=1$). Secondly, we choose the vector $T(e_2)$ to be orthogonal to the orthogonal projection of p on the subspace spanned by e_2, e_3 ($\mathrm{lin}(e_2, e_3)$). Finally, $T(e_3)$ is chosen to be orthogonal to both $T(e_1)$ and $T(e_3)$. It is straightforward to show that this linear transformation fulfils

$$\begin{aligned}
&\sum_{l=1}^{3} e_{k_1}T(e_1)e_{k_2}T(e_1)=0 && k_1<k_2\\
&\sum_{m=1}^{3}\sum_{l=2}^{3} ek_mT(e_1)\prod_{\substack{n=1\\ n\neq m}}^{3} e_{k_n}T(e_1)+3\prod_{n=1}^{3} e_{k_n}T(e_1)=0 && k_1<k_2<k_3.
\end{aligned} \tag{3.24}$$

In addition, T satisfies $T^*=T^{-1}$.

Finally, in view of (3.21) and (3.22), the equations in (3.23)), together with these of (3.24), imply

$$\int_{|v|=\rho} \exp(-\mathrm{i}\rho|p|t\cos\phi)B^n(Tv)\,\mathrm{d}S = (4\pi\rho)\rho^n j_n(\rho|p|t)B^n(\hat{p}) \qquad n=2,3. \tag{3.25}$$

Now, inserting (3.19) and (3.25) into (3.16), and later into (3.14), we finally get lemma 3.2.

Next we will obtain an estimate for $\|G(t,\cdot)f\|_k$ (see equation (6) from [1] for the definition of $\|\ \|_k$). To do this we require some previous inequalities.

Proposition 3.3.

$$|B_t|(p)\leqslant 1 \text{ for all } t\in\mathbb{R}^+,\ p\in\mathbb{R}^3 \tag{3.26}$$

if and only if $\delta\leqslant\sqrt{3}$.

Proof. Let $t\in\mathbb{R}^+$, and $p\in\mathbb{R}^3$. Then, by formulae (3.6) and (3.13), we get $|B_t|^2(\hat{p})\leqslant \Sigma_{n=0}^{3}(\delta^{2n}/n!)j_n^2(t\rho|p|)$. Now, on account of the relationship (3.17) and Lommel's series of squares of Bessel functions (see, e.g., [3], pp 151, 152, equation (2)), we get $\Sigma_{n=0}^{\infty}(3^n/n!)j_n^2(t\rho|p|)\leqslant\Sigma_{n=0}^{\infty}(2n+1)j_n^2(t\rho|p|)=1$. Thus, if $\delta^2\leqslant 3$, then $|B_t(p)|\leqslant 1$ for all $t\in\mathbb{R}^+$, $p\in\mathbb{R}^3$. Conversely, if $\delta>\sqrt{3}$, on account of

$$|B_t(\hat{p})|^2\geqslant\sum_{n=0}^{1} j_n^2(t\rho|p|)(|B|^n(p))^2 = j_0^2(t\rho|p|)+\delta^2 j_1^2(t_\rho|p|) \tag{3.27}$$

(see (3.4)) and the relationship $\lim_{z\to 0}(1-j_0^2(z))/j_1^2(z)=3$, we get $t\in\mathbb{R}^+$, $p\in\mathbb{R}^3$ such that $|B_t(p)|>1$.

Remark 3.5. Note that, if $\pi=\{t_0,\ldots,t_l\}$ is a partition of $[0,s]$, which satisfies $t_l-t_{l-1}=t_{l-1}-t_{l-2}=\ldots=t_1-t_0=s/l=\varepsilon(\pi)<1/\rho$ (i.e. π is a uniform partition), and $\tau\varepsilon(t_k,t_{k+1})$ for some $1\leqslant k\leqslant l$, then, by (3.13) and (3.27), we get

$$\begin{aligned}\|E(\pi,\tau)f\|^2&=\|\,|B|_{\tau-t_k}|B|_{t_k-t_{k-1}}\cdots|B|_{t_1-t_0}f\|^2\\&=\|\,|B|_{\tau-t_k}(|B|_{\varepsilon(\pi)})^k\hat f\|^2\\&\geqslant\|(j_0^2+\delta^2j_1^2)^{1/2}[(\tau-t_k)\rho|\cdot|](j_0^2+\delta^2j_1^2)^{k/2}(\rho\varepsilon(\pi)|\cdot|)\hat f\|^2\\&\geqslant(\rho\varepsilon(\pi))^{-3}\int_{|p|<\eta}(j_0^2+\delta^2j_1^2)^{1/2}[(\tau-t_k)|p|/\varepsilon(\pi)]\\&\quad\times(j_0^2+\delta^2j_1^2)^{k/2}(|p|)|\hat f(p/\rho\varepsilon(\pi))|^2\,\mathrm{d}p\end{aligned}\tag{3.28}$$

(see (3.13)) where η is any positive number which we now determine. Because the derivative of $j_0^2+\delta^2j_1^2$ at zero is positive, by the continuity of its derivative, there is a positive number, which we denote by η, such that the former is an increasing function in $[0,\eta]$. Thus, from (3.27) we obtain

$$\begin{aligned}\|E(\pi,\tau)f\|^2&\geqslant(\rho\varepsilon(\pi))^{-3}\int_{|p|<\eta}(j_0^2+\delta^2j_1^2)^k(|p|)|\hat f(p/\rho\varepsilon(\pi))|^2\,\mathrm{d}p\\&\geqslant(j_0^2+\delta^2j_1^2)^k(\eta/2^{1/3})\int_{\eta\varepsilon(\pi)/2^{1/3}\leqslant|p|\leqslant\varepsilon(\pi)\eta}|\hat f(p)|\,\mathrm{d}p.\end{aligned}\tag{3.29}$$

Now, by means of proposition 3.3, formula (2.5), lemma 3.1 and the Plancherel theorem, we get lemma 3.3.

Lemma 3.3. If $\delta\leqslant 3$ then the operators $G(t,\cdot)$, $E(\pi,\tau)$ are bounded. Their norm is less than 1, i.e.

$$\|G(t,\cdot)f\|\leqslant\|f\|\qquad\|E(\pi,\tau)f\|\leqslant\|f\|\qquad\text{for all } f\in L_2.\tag{3.30}$$

Remark 3.6. The inequalities in the last lemma are as strong as equations (12) and (13) from [1]. For instance, if ϕ, a_k are bounded functions in $C^m(\Omega)$ and have all derivatives up to order m bounded, then the first inequality of (3.30) implies that

$$\|E(\pi,\tau)f\|_m\leqslant K\|f\|_m\qquad\text{for all } f\in C_0^m(\Omega)\tag{3.31}$$

where K is the same function as in lemma 2 of [1]. This can be proved, proceeding as in the proof of lemma 2 of [1] provided that we replace, respectively, A and g by A^j and g^j in formula (2.1), which we have defined in this way. To obtain A^j, $j=1,2,3$, we change the sign in the exponentials of A as in (11) from [1]. Now, if $j=4,5,6$, we define $g^j=v_{j-3}g(g^j=g$ for $j=1,2,3$ and $A^j=A$ for $j=4,5,6)$. Then, we obtain the operators $G^j(t,y)$.

Remark 3.7. Note that formula (3.14) implies that

$$\begin{aligned}&\|(G^j(t,\cdot)-G^j(t_0,\cdot)f\|\\&\quad\leqslant\|(A^j(t,\cdot)-A^j(t_0,\cdot))G(t,\cdot)f\|\\&\quad+\left\|\left(\int_{|v|=\rho}[\exp(-v(\cdot)t)-\exp(-v(\cdot)t_0)]g(v)\,\mathrm{d}S\right)\hat f\right\|\qquad f\in L_2.\end{aligned}\tag{3.32}$$

Lemma 3.4. The maps $t \mapsto G^j(t, \cdot)$ are continuous on $[0, +\infty)$ in the norm of operators, provided that ϕ, a_k are bounded functions and $\delta \leq 3$.

Proof. Fix $t_0 \in [0, +\infty)$. Let $t \in \mathbb{R}^+$, $f \in L_2$. Because ϕ, a_k are bounded, from (3.30), we, for each positive number, can find a positive integer N, which does not depend on t and t_0, such that

$$\int_{|p|>N} |A(t,p) - A(t_0,p)|^2 |G(t,p)f(p)|^2 \, \mathrm{d}p \leq \eta^2/4.$$

On the other hand, since $(\lambda, z_0, z_1, z_2, z_3) \mapsto e^\lambda e^{z0} \Pi_{k=1}^3 \exp(z_k \alpha^k)$ is a continuous function over $\mathbb{C}^5$ (and thus uniformly continuous on any compact set), the boundedness of ϕ, a_k and (3.30) permits us to find, for each positive number η, a positive number r, such that if $|t - t_0| < r$ then

$$\int_{|p|\leq N} |A(t,p) - A(t_0,p)|^2 |G(t,p)f|^2 \, \mathrm{d}p \leq \varepsilon^2/4 \|G(t,\cdot)f\|^2 \leq (\varepsilon^2/4)\|f\|^2.$$

Thus, $\|(A^j(t,\cdot) - A^j(t_0,\cdot))G(t,\cdot)f\| \leq (\varepsilon/2)\|f\| + \eta/2$. To estimate the second term on the right-hand side of (3.32), we proceed as in the first estimation. Thus, from (3.32), we find that $\|G(t,\cdot) - G(t_0,\cdot)f\| \leq \varepsilon\|f\| + \eta$, provided that $|t - t_0| < r$. Hence, we conclude the lemma.

Remark 3.8. Note that the definition of $G^j(t,\cdot)$ implies that $G^j(0,\cdot) = I =$ identity operator, for $j = 0, \ldots, 3$ and $G^j(0,\cdot) = \alpha_{j-3}$ (the operator defined by the matrix α_{j-3}; see remark 3.2).

Remark 3.9. Note that lemma 3.2, together with (3.17), (3.18) and (3.25), implies that

$$G(t,\cdot)f = (1 + b^2 t^2)^{-1} A \mathscr{F}^{-1}(A_t(0)) + \mathrm{o}(t) \qquad f \in C_0(\Omega) \tag{3.33}$$

where $A_t(p)$ is the amplification matrix defined by Jacobson [2] in equation (18). Also observe that, since the amplification matrix can be interpreted as a mean amplitude for a set consisting of all polygonal paths of one segment, we can interpret formula (3.33) in the following way. In an external electromagnetic potential $a = (a_1, a_2, a_3)$ the mean amplitude for each set consisting of all polygonal paths of one segment is multiplied by a matrix A (see equation (34) from [2]).

Remark 3.10. Note that, if $f \in C_0(\Omega)$, then we can differentiate under the integral in (3.14) to obtain

$$\partial A^{-1}(t,y)G(t,y)f/\partial t = -\mathrm{i}\mathscr{F}^{-1}\left[\left(\int_{|v|=\rho} (vp)\exp[-(vp)t]g(v)\,\mathrm{d}S\right)\hat{f}(\rho)\right](y) \tag{3.34}$$

for all $t \in \mathbb{R}^+$, $y \in \mathbb{R}^3$.

Remark 3.11. Note that, if $f \in C_0(\Omega)$, the latter remark implies that $\partial G(t,y)f/\partial t$ exists. In addition

$$\partial G(t,y)f/\partial t = A(t,y)\partial A^{-1}(t,y)/\partial t + \partial A(t,y)/\partial t G(t,y)f. \tag{3.35}$$

Thus, if ϕ, a_k and their first derivatives are bounded from lemma 3.3, we conclude that $\partial G(t,\cdot)f/\partial t \in L_2$.

Remark 3.12. Note that, if $\tau \in (t_k, t_{k+1})$ where $t_k, t_{k+1} \in \pi$, formula (3.14), together with (3.34), implies that

$$E(\pi, \tau + t)f = G(\tau + t, \cdot)E(\pi, t_k)f \qquad t \in (t_k - \tau, t_{k+1} - \tau). \tag{3.36}$$

Lemma 3.5. If $\delta \leqslant 3$, $f \in C_0(\Omega)$ and ϕ, a_k and their first derivatives are bounded then the function $t \mapsto E(\pi, \tau)f$ is differentiable as a function from $\mathbb{R}$ to L_2.

Proof. In view of remarks 2.2 and 3.11, it is sufficient to prove that the function $t \mapsto G(t, \cdot)f$ is differentiable as a function from $\mathbb{R}$ to L_2. Fix $f \in C_0(\Omega)$, $t \in \mathbb{R}^+$. Let $u \in \mathbb{R}^+$. By means of formula (3.14), together with (3.34) and (3.35), it follows that

$$\begin{aligned}
&\|[G(t+u, \cdot) - G(t, \cdot)]/u - [\partial G(t, \cdot)f/\partial t]f\| \\
&\qquad \leqslant \|[(A(t+u), \cdot) - A(t, \cdot))/u - \partial A(t, \cdot)/\partial t]G(t+u, \cdot)f\| \\
&\qquad\quad + \left\| \left(\int_{|v|=\rho} \{[1 - \exp(-\mathrm{i}(\cdot)vt)]/\mathrm{i}(\cdot)vt - 1\} vpg(v) \, \mathrm{d}S \right) f \right\| \\
&\qquad\quad + \|(A^{-1}(t+u, \cdot) - A^{-1}(t, \cdot))G(t+u, \cdot)f\| + \|G(t+u, \cdot) - G(t, \cdot)\| \, \|f\|.
\end{aligned}$$

Because of the boundedness of ϕ, a_k and their first derivatives, from (3.30), the hypotheses of the dominant convergence theorem of Lebesgue are satisfied. Thus, from lemma 3.4, we prove lemma 3.5.

Remark 3.13. Note that, by means of the Baker-Cambell-Hausdorft formula (see, e.g. [5] pp 120, 144),

$$G(t, \cdot) = \exp\left(-\mathrm{i} \sum_{k=1}^{3} t a_k(y) \alpha^k \right) G_0(t, \cdot) \tag{3.37}$$

for short 'time' t. Here $G_0(t, \cdot)$ was obtained by putting $a_k = 0$ in (2.1). Also observe that, because the factor of G_0 is an isometry in $\mathbb{C}^4$ (with Euclidian norm), formula (2.1) can be replaced by that of (3.37) without changing any result obtained in this paper.

4. Main results

We now prove that our approximation converges to the solution (in the L_2-norm) if $\rho \geqslant \sqrt{3}\, c$, c being the speed of light. We can now do this as in [1]. Thus, we will not give complete proofs of the two following results.

Theorem 4.1. Suppose ϕ, a_k and their first derivatives are bounded. Moreover, assume there exists a sequence of partitions $\{\pi_n\}$; with $\varepsilon(\pi_n) \to 0$, and an operator $E(\tau)$ such that, for each $f \in C_0^1(\mathbb{R}^3)$,

$$\|(E(\pi_n, \tau) - E(\tau))f\| \to 0 \qquad \text{when } \varepsilon(\pi_n) \to 0 \tag{4.1}$$

uniformly on $\tau \in [0, s]$. Then, if $\delta \leqslant 3$, $E(\tau)$ is the evolution operator for the Dirac equation (see equation (1) from [1]).

Sketch of the proof. Fix $f \in C_0^1(\mathbb{R}^3)$. Let $\pi = \{t_0, \ldots, t_{l_n}\}$ be a partition of $[0, s]$. Hence there is a t_{k_n} such that $\tau \in (t_{k_n}, t_{k_{n+1}}]$. Now, the definition of the operator $G^j(t, \cdot)$ together with (3.14) and equation (9) from [1] permits us to proceed as in the proof of theorem 3 from [1]. Thus, we obtain

$$
\begin{aligned}
&\partial E(\pi_n, \tau)f/\partial\tau \\
&\quad = HE(\pi_n, \tau)f - \mathrm{i}(\tau - t_{k_n}) \left[\sum_{k=1}^{3} \alpha^k \left(\partial\phi G(\tau - t_{k_n}, y)/\partial y^k \right.\right. \\
&\qquad \left.\left. - \sum_{j=1}^{3} \alpha^j \partial a_j G(\tau - t_{k_n}, y) \right) E(\pi_n, t_{k_n})f \right] \\
&\qquad + \mathrm{i} \sum_{k=1}^{3} \alpha^k a_k(y)(G^k(\tau - t_{k_n}, y) - G(\tau - t_{k_n}, y)E(\pi_n, t_{k_n})f \\
&\qquad - \sum_{k=1}^{3} [G^{k+3}(\tau - t_{k_n}, y) - \alpha^k G(\tau - t_{k_n}, y)]\partial E(\pi_n, t_{k_n})f/\partial x^k
\end{aligned}
\tag{4.2}
$$

where H is the right-hand side of the Dirac equation (see equation (1) from [1]). Now, from (3.31), (4.1) and (4.2), lemma 3.4, together with remark 3.8, implies that

$$
\lim_{n\to\infty} \partial E(\pi_n, \tau)f/\partial\tau = -HE(\tau)f \tag{4.3}
$$

uniformly for $[0, s]$. Due to lemma 3.5 it is possible to interchange the limit and differentiation in (3.3). Theorem 4.1 now follows.

Now we can conclude the convergence of our approximation in the same way as in the proof of theorem 2 from [1] (see lemma 3.3, inequality (3.31), remark 2.4, equation (4.3) and theorem 4.1).

Theorem 4.2. Suppose that ϕ, a_k are bounded functions in $C^2(\mathbb{R}^3)$ and have their first derivatives bounded. If, in addition, $\delta \leqslant 3$, then for each $f \in L_2(\mathbb{R}^3)$, the approximations $E(\pi_n, \tau)$ converge in L_2-norm to $E(\tau)f$, uniformly for $\tau \in [0, s]$, when $\varepsilon(\pi_n)$ converges to zero and where $E(\tau)$ turns out to be the evolution operator for the Dirac equation (see equation (1) from [1]).

The next result proves that there is a minimum allowable 'speed' ρ.

Theorem 4.3. Suppose $\delta > 3$. If, in addition, we assume that ϕ, a_k are constant functions, then there exists a sequence of partitions $\{\pi_n\}_n$ of $[0, s]$ with $\lim_{n\to\infty} \varepsilon(\pi_n) = 0$ such that $\lim_{n\to\infty} \|E(\pi_n, \tau)f\| = \infty$ for all $\tau \in [0, s]$ and $f \in L_2$ with $\inf_{|p|<\eta} |\hat{f}(p)| > 0$ for some positive number η.

Proof. Fix $\varepsilon > 3$ and $f \in L_2$. Let $\tau \in (0, s]$. Assume that $\pi_n = \{t_0, \ldots, t_{l_n}\}$ is a partition of $[0, s]$. Then there exists t_{k_n} such that $\tau \in (t_{k_n}, t_{k_{n+1}}]$. If, in addition, we assume that $t_{l_n} - t_{l_{n-1}} = t_{l_{n-1}} - t_{l_{n-2}} = \ldots = t_{ln} - t_0 = s/n = \varepsilon(\pi_n) < 1/\rho$ and $\inf_{|p|<\eta} |\hat{f}(p)| > 0$ for some positive number η. Then, by remark 3.5 and (3.28), we obtain $\|E(\pi, \tau)f\|^2 \geqslant Ct_{k_n}^3 r^{kn}/k_n^3$ where $r > 1$, $C = \frac{2}{3}\eta^3 \inf_{|p|<\eta} |\hat{f}(p)| > 0$. Hence, since $\lim_{n\to\infty} t_k(1/s) \lim_{n\to\infty} k_n/n = \tau$ implies that $\lim_{n\to\infty} k_n = \infty$, we can conclude that $\lim_{n\to\infty} \|E(\pi, \tau)f\| = \infty$.

Remark 4.1. Note that all results of this paper can be extended to the d-dimensional Dirac equation. For instance, the operator $G(t,\cdot)$ should be defined by

$$G(t,y)f = A(S)^{-1}A\int_{|v|=\rho} gf(y-tv)\,\mathrm{d}S$$

where $\mathrm{d}S$ is the area element of a d-dimensional sphere of radius ρ. Formula (3.8), in this case, would be written as

$$G(t,y)f = A\mathscr{F}^{-1}\left(\Gamma(\nu+1)(2/t\rho|p|)^{\nu}\sum_{n=0}^{d}(-\mathrm{i}\delta)^{n}j_{n+\nu}(t\rho|p|)|B|^{n}(\hat{p})\right).$$

Here $\nu=(d-2)/2$ and $j_{n+\nu}$, $n=1,2,\ldots,d$ are Bessel functions of the first kind. In this case, the critical 'speed' is $\sqrt{d}\,c$ which can be obtained by Lommel's series of squares of Bessel functions.

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